

Dual morse index estimates and application to Hamiltonian systems with P-boundary conditions

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Abstract

In this paper, we study the multiplicity of Hamiltonian systems with P-boundary conditions.

Keywords: Maslov P -index, dual Morse index, Hamiltonian systems, P-boundary conditions

2000 Mathematics Subject Classification: 58F05, 58E05, 34C25, 58F10

1 Introduction and main result

We consider the solutions of nonlinear Hamiltonian systems with P -boundary condition

$$\begin{cases} \dot{x} = JH'(t, x), \quad \forall x \in \mathbb{R}^{2n}, \\ x(1) = Px(0), \end{cases} \quad (1.1)$$

where $P \in Sp(2n)$ satisfies $P^T P = I_{2n}$, $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the standard symplectic matrix, I_n, I_{2n} are the identity matrices on \mathbb{R}^n and \mathbb{R}^{2n} , n is the positive integer. The Hamiltonian function $H \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ satisfies the following conditions:

(H) $H(t+1, Px) = H(t, x)$, $\forall (t, x) \in \mathbb{R} \times \mathbb{R}^{2n}$;

(H₀) $H'(t, 0) \equiv 0$;

(H_∞) There two continuous symmetric matrix functions $B_j(t)$, $j = 1, 2$ satisfying $P^T B_j(t+1)P = B_j(t)$, $i_P(B_1) = i_P(B_2)$ and $\nu_P(B_2) = 0$ such that

$$B_1(t) \leq H''(t, x) \leq B_2(t), \quad \forall (t, x) \text{ with } |x| \geq r \text{ for some large } r > 0.$$

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Let $\mathfrak{L}_s(\mathbb{R}^{2n})$ denotes all symmetric real $2n \times 2n$ matrices. For $A, B \in \mathfrak{L}_s(\mathbb{R}^{2n})$, $A \geq B$ means that $A - B$ is a semipositive definite matrix, and $A > B$ means that $A - B$ is a positive definite matrix.

Theorem 1.1. *Suppose that $P \in Sp(2n)$ satisfies $P^T P = I_{2n}$, H satisfies conditions (H) , (H_0) , (H_∞) . Suppose $JB_1(t) = B_1(t)J$ and $B_0(t) = H''(t, 0)$ satisfying one of the twisted conditions*

$$B_1(t) + lI_{2n} \leq B_0(t) \quad (1.2)$$

$$B_0(t) + lI_{2n} \leq B_1(t) \quad (1.3)$$

for some constant $l \geq 2\pi$. Then the system (1.1) possesses at least one non-trivial P -solution. Furthermore, if $\nu_P(B_0) = 0$, the system (1.1) possesses at least two non-trivial P -solutions.

A solution $(1, x)$ of the problem (1.1) is called P -solution of the Hamiltonian systems. It is a kind of generalized periodic solution of Hamiltonian systems. The problem (1.1) has relation with the closed geodesics on Riemannian manifold (cf.[14]) and symmetric periodic solution or the quasi-periodic solution problem (cf.[15]). In addition, C. Liu in [22] transformed some periodic boundary problem for asymptotically linear delay differential systems and some asymptotically linear delay Hamiltonian systems to P -boundary problems of Hamiltonian systems as above, we also refer [3, 11, 16, 17] and references therein for the background of P -boundary problems in N -body problems.

We briefly review the general study of the problem (1.1). C. Liu (cf. [21]) used the Maslov P-index theory, the Poincaré polynomial of the Conley homotopic index of isolated invariant set and saddle point reduction method to study (1.1) with asymptotically linear condition. At the same time, Y. Dong (cf.[7]) combined dual Morse index with Maslov P-index to study the existence and multiplicity of (1.1) under convexity condition. Since then, many papers appeared about the study of P-boundary problems (cf.[8, 22, 23, 26, 27, 28]). Besides, Y. Dong defined another dual Morse index to study the Bolza problem in [6]. In this paper, we develop l -dual Morse index about P -boundary problem which is similar to [6] and then use it to discuss the existence and multiple solutions of (1.1). The dual Morse index theory for periodic boundary condition was studied by Girardi and Matzeu [12] for superquadratic Hamiltonian systems and by C. Liu in [19] for subquadratic Hamiltonian systems. This theory is an application of the Morse-Ekeland index theory (cf.[10]). The index theory for convex Hamiltonian systems was established by I. Ekeland (cf.[10]), whose works are of fundamental importance in the study of convex Hamiltonian systems.

This paper is divided into 4 sections. In Section 2, we recall Maslov P -index and study some properties. In Section 3, we use the method in [6, 20] to develop l -dual Morse index which is suitable for P -boundary problems and discuss the relationship with Maslov P -index. Base on the study of Section 2 and 3, in Section 4, we look for P -solution of (1.1) under twisted conditions and prove Theorem 1.1.

2 Some properties of the Maslov P-index theory

Maslov P-index was first studied in [7] and [21] independently for any symplectic matrix P with different treatment, it was generalized by C. Liu and the author in [25, 26]. And then

C. Liu used relative index theory to develop Maslov P-index in [23] which is consistent with the definition in [25, 26]. In fact, when the symplectic matrix $P = \text{diag}\{-I_{n-\kappa}, I_\kappa, -I_{n-\kappa}, I_\kappa\}$, $0 \leq \kappa \in \mathbb{Z} \leq n$, the (P, ω) -index theory and its iteration theory were studied in [8] and then be successfully used to study the multiplicity of closed characteristics on partially symmetric convex compact hypersurfaces in \mathbb{R}^{2n} . Here we use the notions and results in [23, 25, 26].

For $P \in Sp(2n)$, $B(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$ and satisfies $P^T B(t+1)P = B(t)$. If γ is the fundamental solution of the linear Hamiltonian systems

$$\dot{y}(t) = JB(t)y, \quad y \in \mathbb{R}^{2n}. \quad (2.1)$$

Then the Maslov P -index pair of γ is defined as a pair of integers

$$(i_P(B), \nu_P(B)) \equiv (i_P(\gamma), \nu_P(\gamma)) \in \mathbb{Z} \times \{0, 1, \dots, 2n\}, \quad (2.2)$$

where i_P is the index part and

$$\nu_P = \dim \ker(\gamma(1) - P)$$

is the nullity. We also call (i_P, ν_P) the Maslov P-index of $B(t)$, just as in [23, 25, 26]. If x is a P -solution of (1.1), then the Maslov P-index of the solution x is defined to be the Maslov P-index of $B(t) = H''(x(t))$ and denoted by $(i_P(x), \nu_P(x))$.

The Hilbert space $W^{1/2,2}([0, 1], \mathbb{R}^{2n})$ consists of all the elements of $z \in L^2([0, 1], \mathbb{R}^{2n})$ satisfying

$$z(t) = \sum_{j \in \mathbb{Z}} \exp(2j\pi t J) a_j, \quad \sum_{j \in \mathbb{Z}} (1 + |j|) a_j^2 < \infty, \quad a_j \in \mathbb{R}^{2n}.$$

For $P \in Sp(2n)$, we define

$$W_P = \{z \in W^{1/2,2}([0, 1], \mathbb{R}^{2n}) \mid z(t+1) = Pz(t)\},$$

it is a closed subspace of $W^{1/2,2}([0, 1], \mathbb{R}^{2n})$ and is also a Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ as in $W^{1/2,2}([0, 1], \mathbb{R}^{2n})$.

Let $\mathfrak{L}_s(W_P)$ and $\mathfrak{L}_c(W_P)$ denote the space of the bounded selfadjoint linear operator and compact linear operator on W_P . We define two operators $A, B \in \mathfrak{L}_s(W_P)$ by the following bilinear forms:

$$\langle Ax, y \rangle = \int_0^1 (-J\dot{x}(t), y(t)) dt, \quad \langle Bx, y \rangle = \int_0^1 (B(t)x(t), y(t)) dt. \quad (2.3)$$

Then $B \in \mathfrak{L}_c(W_P)$ (cf. [30]). Using the Floquet theory we have

$$\nu_P(B) = \dim \ker(A - B). \quad (2.4)$$

Suppose that $\dots \leq \lambda_{-j} \leq \dots \leq \lambda_{-1} < 0 < \lambda_1 \leq \dots \leq \lambda_j \leq \dots$ are all nonzero eigenvalues of the operator A (count with multiplicity), correspondingly, g_j is the eigenvector of λ_j satisfying $\langle g_j, g_i \rangle = \delta_{ji}$. We denote the kernel of the operator A by W_P^0 , specially it is exactly the space $\ker_{\mathbb{R}}(P - I)$. For $m \in \mathbb{N}$, we define a finite dimensional subspace of W_P by

$$W_P^m = W_m^- \oplus W_P^0 \oplus W_m^+$$

with $W_m^- = \{z \in W_P | z(t) = \sum_{j=1}^m a_{-j} g_{-j}(t), a_{-j} \in \mathbb{R}\}$ and $W_m^+ = \{z \in W_P | z(t) = \sum_{j=1}^m a_j g_j(t), a_j \in \mathbb{R}\}$.

We suppose P_m be the orthogonal projections $P_m : W_P \rightarrow W_m^m$ for $m \in \mathbb{N} \cup \{0\}$. Then $\{P_m | m = 0, 1, 2, \dots\}$ be the Galerkin approximation sequence respect to A .

For $S \in \mathfrak{L}_s(W_P)$, we denote by $M^*(S)$ the eigenspaces of S with eigenvalues belonging to $(0, +\infty)$, $\{0\}$ and $(-\infty, 0)$ with $*$ = +, 0 and $*$ = -, respectively. Similarly, for any $d > 0$, we denote by $M_d^*(S)$ the d -eigenspaces of S with eigenvalues belonging to $[d, +\infty)$, $(-d, d)$ and $(-\infty, -d]$ with $*$ = +, 0 and $*$ = -, respectively. We denote $m^*(S) = \dim M^*(S)$, $m_d^*(S) = \dim M_d^*(S)$ and $S^\sharp = (S|_{I_m S})^{-1}$.

Theorem 2.1. Suppose $B(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$ and satisfies $P^T B(t+1)P = B(t)$ with the Maslov P -index $(i_P(B), \nu_P(B))$, for any constant $0 < d < \frac{1}{4} \|(A - B)^\sharp\|^{-1}$, there exists an $m_0 > 0$ such that for $m \geq m_0$, there holds

$$\begin{aligned} m_d^+(P_m(A - B)P_m) &= m + \dim \ker_{\mathbb{R}}(P - I_{2n}) - i_P(B) - \nu_P(B), \\ m_d^-(P_m(A - B)P_m) &= m + i_P(B), \\ m_d^0(P_m(A - B)P_m) &= \nu_P(B), \end{aligned} \tag{2.5}$$

where B be the operator defined by (2.10) corresponding to $B(t)$.

As a direct consequence, we have the following monotonicity result.

Corollary 2.2. Suppose $B_j(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$, $j = 1, 2$ with $P^T B_j(t + \tau)P = B_j(t)$ satisfy

$$B_1(t) < B_2(t), \text{ i.e., } B_2(t) - B_1(t) \text{ is positive definite for all } t \in [0, \tau]. \tag{2.6}$$

There there holds

$$i_P(B_1) + \nu_P(B_1) \leq i_P(B_2). \tag{2.7}$$

Proof. Let $\Gamma = \{P_m\}$ be the approximation scheme with respect to the operator A . Then by (2.5), there exists $m_0 > 0$ such that if $m \geq m_0$, there holds

$$\begin{aligned} m_d^-(P_m(A - B_1)P_m) &= m + i_P(B_1), \\ m_d^-(P_m(A - B_2)P_m) &= m + i_P(B_2), \\ m_d^0(P_m(A - B_1)P_m) &= \nu_P(B_1), \end{aligned}$$

when $0 < d < \frac{1}{2} \|B_2 - B_1\|$. Since $A - B_2 = (A - B_1) - (B_2 - B_1)$ and $B_2 - B_1$ is positive definite in W_P^m and $\langle (B_2 - B_1)x, x \rangle \geq 2d\|x\|$. Hence we have $\langle P_m^-(A - B_1)P_m x, x \rangle \leq -d\|x\|$ with

$$x \in M_d^-(P_m^-(A - B_1)P_m) \oplus M_d^0(P_m^-(A - B_1)P_m).$$

It implies that $m + i_P(B_1) + \nu_P(B_1) \leq m + i_P(B_2)$. □

Remark 2.3. From the proof of Corollary 2.2, it is easy to show that if $B_1(t) \leq B_2(t)$ for all $t \in [0, \tau]$,

$$i_P(B_1) \leq i_P(B_2), \quad i_P(B_1) + \nu_P(B_1) \leq i_P(B_2) + \nu_P(B_2). \tag{2.8}$$

Definition 2.4. For $P \in Sp(2n)$, suppose $B_j(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$, $j = 1, 2$ satisfies $B_j(t+1) = (P^{-1})^T B_j(t) P^{-1}$ and $B_1(t) < B_2(t)$ for all $t \in [0, 1]$, we define

$$I_P(B_1, B_2) = \sum_{s \in [0, 1]} \nu_P((1-s)B_1 + sB_2).$$

Theorem 2.5. For $P \in Sp(2n)$, suppose $B_j(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$, $j = 1, 2$ satisfies $B_j(t+1) = (P^{-1})^T B_j(t) P^{-1}$ and $B_1(t) < B_2(t)$ for all $t \in [0, 1]$, there holds

$$I_P(B_1, B_2) = i_P(B_2) - i_P(B_1).$$

Hence we call $I_P(B_0, B_1)$ the relative P -index of the pair (B_1, B_2) .

In [23], C. Liu proved that $m_d^0(P_m(A-B)P_m)$ eventually becomes a constant independent of m and for large m , there holds

$$m_d^0(P_m(A-B)P_m) = m^0(A-B). \quad (2.9)$$

Hence

$$m_d^-(P_m(A-B)P_m) = m^-(P_m(A-B)P_m). \quad (2.10)$$

And further, the difference of the d -Morse indices $m_d^-(P_m(A-B)P_m) - m_d^-(P_m A P_m)$ eventually becomes a constant independent of m for large m , where $d > 0$ is determined by the operators A and $A - B$. Then he defined the relative index by

$$I(A, A-B) = m_d^-(P_m(A-B)P_m) - m_d^-(P_m A P_m), \quad m \geq m^*, \quad (2.11)$$

and got the following important results.

Theorem 2.6. Suppose $B(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$ satisfies $B(t+1) = (P^{-1})^T B(t) P^{-1}$, there holds

$$I(A, A-B) = i_P(B). \quad (2.12)$$

Lemma 2.7. Suppose $B_j(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$, $j = 1, 2$ satisfy $P^T B_j(t+1) P = B_j(t)$ and $B_1(t) < B_2(t)$ for all $t \in \mathbb{R}$, there holds

$$i_P(B_2) - i_P(B_1) = \sum_{s \in [0, 1]} \nu_P((1-s)B_1 + sB_2). \quad (2.13)$$

Theorem 2.8. The Maslov P -index defined by (2.2) as in [25], the relative P -index defined by Definition 2.4 have the following properties:

- (1) For $P \in Sp(2n)$, $B_j(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$, $j = 1, 2, 3$ satisfy $P^T B_j(t+1) P = B_j(t)$ and $B_1(t) < B_2(t) < B_3(t)$ for all $t \in \mathbb{R}$, we have

$$I_P(B_1, B_2) + I_P(B_2, B_3) = I_P(B_1, B_3).$$

(2) For $P \in Sp(2n)$ with $P^T P = I_{2n}$, $B(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$ satisfies $P^T B(t+1)P = B(t)$, there exist $s_0 > 0$ such that for any $s \in (0, s_0]$, we have

$$\begin{aligned}\nu_P(B + sI_{2n}) &= 0 = \nu_P(B - sI_{2n}), \\ i_P(B - sI_{2n}) &= i_P(B), \\ i_P(B + sI_{2n}) &= i_P(B) + \nu_P(B).\end{aligned}$$

In particular, if $\nu_P(B) = 0$, we have $i_P(B + sI_{2n}) = i_P(B)$ for $s \in (0, s_0]$.

Proof. (1) follows from Theorem 2.5 immediately.

From Theorem 2.5, we have $i_P(B + I_{2n}) = I_P(B, B + I_{2n}) - i_P(B)$. By Lemma 2.7, we see that $I_P(B, B + I_{2n}) = \sum_{s \in [0,1)} \nu_P(B + sI_{2n})$ is finite. So there is some s_0 such that $\nu_P(B + sI_{2n}) = 0$ for $s \in (0, s_0]$, and

$$i_P(B + sI_{2n}) = i_P(B) + \sum_{\lambda \in [0,1)} \nu_P(B + \lambda sI_{2n}) = i_P(B) + \nu_P(B). \quad (2.14)$$

Similarly, $i_P(B - I_{2n}, B) = i_P(B) - i_P(B - I_{2n}) = \sum_{s \in [0,1)} \nu_P(B - (1-s)I_{2n})$ is finite, so there is some s_0 such that $\nu_P(B - sI_{2n}) = 0$ for $s \in (0, s_0]$, and

$$i_P(B - sI_{2n}) = i_P(B) - \sum_{\lambda \in [0,1)} \nu_P(B - (1-\lambda)sI_{2n}) = i_P(B). \quad (2.15)$$

If $\nu_P(B) = 0$, by (2.14) we have $i_P(B + sI_{2n}) = i_P(B)$ for $s \in (0, s_0]$.

□

3 Dual morse index theory for linear Hamiltonian systems with P -boundary conditions

Recall that the Hilbert space $W_P = \{z \in W^{1/2,2}([0,1], \mathbb{R}^{2n}) \mid z(t+1) = Pz(t)\}$ with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Let $L_P = \{z \in L^2([0,1], \mathbb{R}^{2n}) \mid z(t+1) = Pz(t)\}$ with norm $\|\cdot\|_2$ and inner product $\langle \cdot, \cdot \rangle_2$. By the well-known Sobolev embedding theorem, the embedding $j : W_P \rightarrow L_P$ is compact. For $P \in Sp(2n)$ with $P^T P = I_{2n}$, we define an operator $A : L_P \rightarrow L_P$ with domain W_P by $A = -Jd/dt$. The spectrum of A is isolated. Let $l \notin \sigma(A)$ be so large such that $B(t) + lI_{2n} > 0$. Then the operator $\Lambda_l = A + lI_{2n} : W_P \rightarrow L_P$ is invertible and its inverse is compact. We define a quadratic form in L_P by

$$Q_{l,B}^*(u, v) = \int_0^1 (C_l(t)u(t), v(t)) - (\Lambda_l^{-1}u(t), v(t)) \quad \forall u, v \in L_P, \quad (3.1)$$

where $C_l(t) = (B(t) + lI_{2n})^{-1}$. Define $Q_{l,B}^*(u) = Q_{l,B}^*(u, u)$. We define the operator $C_l : L_P \rightarrow L_P$ by

$$\langle C_l u, v \rangle_2 = \int_0^1 (C_l(t)u(t), v(t)) dt.$$

Since $C_l(t)$ is positive definite, C_l is an isomorphism and $\langle C_l u, u \rangle_2$ defines a Hilbert space structure on L which is equivalent to the standard one. Endowing L_P with the inner product $\langle C_l u, u \rangle_2$, Λ_l^{-1} is a self-adjoint and compact operator and applying to Λ_l^{-1} the spectral theory of compact self-adjoint operators on Hilbert space, we see there is a basis e_j , $j \in \mathbb{N}$ of L_P , and an eigenvalue sequence $\mu_j \rightarrow 0$ in \mathbb{R} such that

$$\begin{aligned} \langle C_l e_i, e_j \rangle_2 &= \delta_{ij}, \\ \langle \Lambda_l^{-1} e_j, u \rangle_2 &= \langle C_l \mu_j e_j, u \rangle_2, \quad \forall u \in L_P. \end{aligned} \quad (3.2)$$

Hence, expressing any vector $u \in L$ as $u = \sum_{j=1}^{\infty} \xi_j e_j$,

$$\begin{aligned} Q_{l,B}^*(u) &= \int_0^1 (C_l(t)u(t), u(t))dt - \int_0^1 (\Lambda_l^{-1}u(t), v(t))dt \\ &= \sum_{j=1}^{\infty} \xi_j^2 - \sum_{j=1}^{\infty} \mu_j \xi_j^2 = \sum_{j=1}^{\infty} (1 - \mu_j) \xi_j^2. \end{aligned}$$

Define

$$\begin{aligned} L_l^+(B) &= \left\{ \sum_{j=1}^{\infty} \xi_j e_j \mid \xi_j = 0 \text{ if } 1 - \mu_j \leq 0 \right\}, \\ L_l^0(B) &= \left\{ \sum_{j=1}^{\infty} \xi_j e_j \mid \xi_j = 0 \text{ if } 1 - \mu_j \neq 0 \right\}, \\ L_l^-(B) &= \left\{ \sum_{j=1}^{\infty} \xi_j e_j \mid \xi_j = 0 \text{ if } 1 - \mu_j \geq 0 \right\}. \end{aligned}$$

Observe that $L_l^+(B)$, $L_l^0(B)$ and $L_l^-(B)$ are $Q_{l,B}^*$ -orthogonal, and $L_P = L_l^+(B) \oplus L_l^0(B) \oplus L_l^-(B)$. Since $\mu_j \rightarrow 0$ when $j \rightarrow \infty$, all the coefficients $1 - \mu_j$ are positive except a finite number. It implies that both $L_l^+(B)$ and $L_l^0(B)$ are finite subspaces.

Definition 3.1. For $P \in Sp(2n)$ with $P^T P = I_{2n}$, $B(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$ satisfy $P^T B(t+1)P = B(t)$, $l \in \mathbb{R}$ with $B(t) + lI_{2n} > 0$, we define

$$i_l^*(B) = \dim L_l^-(B), \quad \nu_l^*(B) = \dim L_l^0(B). \quad (3.3)$$

We call $i_l^*(B)$ and $\nu_l^*(B)$ the l -dual Morse index and l -dual nullity of B respectively.

Theorem 3.2. Under the conditions of Theorem 2.5 and Definition 3.1, we have

$$\nu_l^*(B) = \nu_P(B), \quad I_P(B_1, B_2) = i_l^*(B_2) - i_l^*(B_1).$$

Proof. We follow the idea in [6] to prove it.

By definitions, $L_l^+(B)$, $L_l^0(B)$ and $L_l^-(B)$ are $Q_{l,B}^*$ -orthogonal, and satisfy $L_P = L_l^+(B) \oplus L_l^0(B) \oplus L_l^-(B)$. For every $u \in L_l^-(B)$, we have

$$Q_{l,B}^*(u, v) = 0, \quad \forall v \in L_P.$$

So

$$C_l(t)u(t) - \Lambda_l^{-1}u(t) = 0.$$

Set $x = \Lambda_l^{-1}u$. Applying $C_l(t) = (B(t) + lI_{2n})^{-1}$ to both sides and using the equalities $\Lambda_l = -J\frac{d}{dt} + l$ and $u = \Lambda_l x$, we obtain

$$-J\dot{x}(t) + lx(t) - (B(t) + lI_{2n})x(t) = 0.$$

That is,

$$\dot{x}(t) = JB(t)x(t). \quad (3.4)$$

Hence $\nu_l^*(B)$ is the dimension of $\ker(\gamma(1) - P)$, where $\gamma(t)$ is the fundamental solution of (3.4) and $\nu_l^*(B) = \nu_P(B)$.

We carry out the proof of the second equality in several steps.

Step 1. We show that if X is a subspace of L_P such that $Q_{l,B}^*(u, u) < 0$ for every $u \in X \setminus 0$, then $\dim X \leq i_l^*(B)$.

In fact, suppose e_1, \dots, e_r be a basis of X , we have the decomposition $e_i = e_i^- + e_i^*$ with $e_i^- \in L_l^-(B)$, $e_i^* \in L_l^+(B) \oplus L_l^0(B)$.

Suppose there exist numbers $\alpha_i \in \mathbb{R}$ which are not all zero, such that $\sum_{i=1}^r \alpha_i e_i^- = 0$.

Set $e = \sum_{i=1}^r \alpha_i e_i$, then $e \in X \setminus 0$ and $Q_{l,B}^*(e, e) < 0$; at the same time, $e = \sum_{i=1}^r \alpha_i e_i^* \in L_l^+(B) \oplus L_l^0(B)$ and $Q_{l,B}^*(e, e) \geq 0$, a contradiction.

So $\{e_i^-\}_{i=1}^r$ is linear independent and $i_l^*(B) \geq r = \dim X$.

Step 2. For $B_j(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$, $j = 1, 2$ satisfy $P^T B_j(t+1)P = B_j(t)$ and $B_1(t) < B_2(t)$. Set $i(\lambda) = i_l^*((1-\lambda)B_1 + \lambda B_2)$ for $\lambda \in [0, 1]$. Then $i(\lambda_2) \geq i(\lambda_1) + \nu(\lambda_1)$.

In fact, set $A_i = (1 - \lambda_i)B_1 + \lambda_i B_2$ for $i = 1, 2$. We only need to prove that

$$Q_{l,A_2}^*(u, u) < 0, \quad \forall u \in L_l^-(A_1) \oplus L_l^0(A_1) \setminus 0.$$

Take any $u = u^0 + u^-$ with $u^0 \in L_l^0(A_1)$, $u^- \in L_l^-(A_1)$. Note that

$$A_2 - A_1 = (\lambda_1 - \lambda_2)(B_1 - B_2) > 0$$

If $u^- \neq 0$, we have

$$\begin{aligned} Q_{l,A_2}^*(u, u) &\leq Q_{l,A_1}^*(u, u) = Q_{l,A_1}^*(u^-, u^-) + Q_{l,A_1}^*(u^0, u^0) \\ &= Q_{l,A_1}^*(u^-, u^-) < 0. \end{aligned}$$

If $u^- = 0$, set $x^0 = \lambda_l^{-1}u^0$, then $u^0 = \lambda_l x^0$ and x^0 is a nontrivial solution of

$$J\dot{x}(t) + A_1(t)x(t) = 0, \quad x(1) = Px(0).$$

So $x^0(t) \neq 0$ for every $t \in [0, 1]$, and $u^0 = (A_1(t) + lI_{2n})x^0(t) \neq 0$ for a.e. $t \in (0, 1)$. Hence

$$\begin{aligned} \frac{1}{\lambda_1 - \lambda_2} Q_{l,A_2}^*(u, u) &= \frac{1}{\lambda_1 - \lambda_2} (Q_{l,A_2}^*(u^0, u^0) - Q_{l,A_1}^*(u^0, u^0)) \\ &= \int_0^1 \left(\frac{(B_2(t) - B_1(t))u^0(t)}{(A_2(t) + lI_{2n})(A_1(t) + lI_{2n})}, u^0(t) \right) dt \\ &= \int_0^1 \left(\frac{(B_2(t) - B_1(t))x^0(t)}{A_2(t) + lI_{2n}}, (A_1(t) + lI_{2n})x^0(t) \right) dt. \end{aligned}$$

If $\lambda_1 = \lambda_2$, we have $A_1(t) = A_2(t)$ and the last integral is

$$\int_0^1 ((B_2(t) - B_1(t))x^0(t), x^0(t))dt > 0.$$

Hence, if λ_2 is close to λ_1 and $\lambda_2 > \lambda_1$, we have $Q_{l,A_2}^*(u, u) < 0$. So for $\lambda_2 > \lambda_1$ and λ_2 is close to λ_1 , we have $i(\lambda_2) \geq i(\lambda_1) + \nu(\lambda_1)$.

Step 3. For any $\lambda \in [0, 1]$, we have $i(\lambda + 0) = i(\lambda) + \nu(\lambda)$, where $i(\lambda + 0)$ is the right limit of $i(\lambda')$ at λ , $\nu_\lambda = \nu_l^*((1 - \lambda)B_1 + \lambda B_2)$.

In fact, we have $i(\lambda) + \nu(\lambda) \leq i(\lambda + 0)$ by Step 2. So we only need to prove that $i(\lambda) + \nu(\lambda) \geq i(\lambda + 0)$. Set $d = i(\lambda + 0)$. There exists $\lambda' > \lambda$ such that $i(s) = d$ and $\nu(s) = 0$ for $s \in (\lambda, \lambda')$. Set $C(s) = ((1 - s)B_1 + sB_2 + lI)^{-1}$. Similar to (3.2), we have

$$\begin{aligned} \langle C(s)e_i^s, e_j^s \rangle_2 &= \delta_{ij}, \\ \langle \Lambda_l^{-1}e_j^s, u \rangle_2 &= \langle C(s)\mu_j^s e_j^s, u \rangle_2, \quad \forall u \in L_P. \end{aligned} \tag{3.5}$$

Since $C(s) \geq (B_2(t) + lI)^{-1}$ for $s \in [0, 1]$, the sequence $\{e_j^s\}$ is bounded in L_P and $\mu_j^s = \langle \Lambda_l^{-1}e_j^s, e_j^s \rangle_2$ is bounded in \mathbb{R} for $j = 1, \dots, d$. So there exist $s_k \in (\lambda, \lambda')$ such that $s_k \rightarrow \lambda + 0$, $e_j^{s_k} \rightarrow e_j$ in L_P , $\mu_j^{s_k} \rightarrow \mu_j$ in \mathbb{R} and $\Lambda_l^{-1}e_j^{s_k} \rightarrow \Lambda_l^{-1}e_j$.

Taking the limit in (3.5) we obtain $\langle C(\lambda)e_i, e_j \rangle_2 = \delta_{ij}$ and $\Lambda_l^{-1}e_j = C(\lambda)\mu_j e_j$ for $j = 1, \dots, d$. Again for $j = 1, \dots, d$, since $i(s) = d$ for $s \in (\lambda, \lambda')$, by definition we have $1 + \mu_j^{s_k} < 0$ and $1/\mu_j^{s_k}$ is bounded in \mathbb{R} . Hence

$$e_j^{s_k} = \frac{1}{\mu_j^{s_k}} C(s_k)^{-1} \Lambda_l^{-1} e_j^{s_k} \rightarrow \frac{1}{\mu_j} C(\lambda)^{-1} \Lambda_l^{-1} e_j = e_j$$

in L_P . It follows that $\{e_i\}_{i=1}^d$ is linearly independent and for every $u = \sum_{j=1}^d \alpha_j e_j$, since $\sum_{j=1}^d \alpha_j e_j^{s_k} \rightarrow u$ in L_P and

$$Q_{l, (1-s_k)B_1 + s_k B_2}^* \left(\sum_{j=1}^d \alpha_j e_j^{s_k}, \sum_{j=1}^d \alpha_j e_j^{s_k} \right) < 0,$$

taking the limit as $s_k \rightarrow \lambda + 0$, we have $Q_{l, (1-\lambda)B_1 + \lambda B_2}^*(u, u) \leq 0$. In a way similar to the proof of Step 1, this implies $i(\lambda) + \nu(\lambda) \geq d = i(\lambda + 0)$.

Step 4. The function $i(\lambda)$ is left continuous for $\lambda \in (0, 1]$ and continuous for $\lambda \in (0, 1)$ with $\nu_\lambda = 0$.

In fact, from Step 2 and 3 we only need to show $i(\lambda) \leq i(\lambda - 0)$. Let e_1, \dots, e_k be a basis of $L^-(\lambda) := L_l^-((1 - \lambda)B_1 + \lambda B_2)$, and

$$S_1 := \{(\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d \mid \sum_{i=1}^k \alpha_i^2 = 1\}.$$

Then

$$\begin{aligned} f(s, \alpha_1, \dots, \alpha_d) &:= Q_{l, (1-s)B_1 + sB_2}^* \left(\sum_{j=1}^d \alpha_j e_j, \sum_{j=1}^d \alpha_j e_j \right) \\ &= \int_0^1 [(((1-s)B_1(t) + sB_2(t) + lI_{2n})^{-1} \sum_{j=1}^d \alpha_j e_j, \sum_{j=1}^d \alpha_j e_j) - (\Lambda_l^{-1} \sum_{j=1}^d \alpha_j e_j, \sum_{j=1}^d \alpha_j e_j)] dt \end{aligned}$$

is continuous in $[0, 1] \times S_1$. Since $f(\lambda, \alpha_1, \dots, \alpha_d) < 0$ for $(\alpha_1, \dots, \alpha_d) \in S_1$, we have $f(s, \alpha_1, \dots, \alpha_d) < 0$ for $(\alpha_1, \dots, \alpha_d) \in S_1$ and s close enough to λ .

From Step 1, we have $i(\lambda) \leq i(s)$ for s close to λ . Hence $i(\lambda) \leq i(\lambda - 0)$. In conclusion,

$$\begin{aligned} i_l^*(B_2) &= i_l^*(B_1) + \sum_{0 \leq \lambda < 1} \nu_l((1-\lambda)B_1 + \lambda B_2) \\ &= i_l^*(B_1) + \sum_{0 \leq \lambda < 1} \nu_P((1-\lambda)B_1 + \lambda B_2) = i_l^*(B_1) + I_P(B_1, B_2). \end{aligned}$$

□

Further, if P satisfies $P^k = I_{2n}$ for some $k \in \mathbb{R}$, we can obtain the specific formula of $i_l^*(B)$ by the method used in [20].

Theorem 3.3. *Suppose that $P \in Sp(2n)$ satisfies $P^T P = I_{2n}$ and $P^k = I_{2n}$ for some $k \in \mathbb{R}$, under the conditions of Theorem 3.2, there holds*

$$\nu_l^*(B) = \nu_P(B), \quad i_l^*(B) = 2mn + i_P(B) - M, \quad (3.6)$$

where M is independent of B and satisfies

$$2n(m-1 - \lfloor \frac{[l/2\pi]}{k} \rfloor) \leq M \leq 2n(m - \lfloor \frac{[l/2\pi]}{k} \rfloor). \quad (3.7)$$

Proof. For $P \in Sp(2n)$ with $P^k = I$, we can regard W_P as

$$W_P = \{z \in W^{1/2,2}(S_k, \mathbb{R}^{2n}) \mid z(t+1) = Pz(t)\}, \quad S_k = \mathbb{R}/k\mathbb{Z},$$

it is a closed subspace of $W^{1/2,2}(S_k, \mathbb{R}^{2n})$ and is also a Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ as in $W^{1/2,2}(S_k, \mathbb{R}^{2n})$.

By a direct computation, we see that $z \in W_P$ iff $z \in W^{1/2,2}(S_k, \mathbb{R}^{2n})$ and a_0 is an eigenvector of the eigenvalue 1 of P and $a_j = \alpha_j + J\beta_j$, $a_{-j} = \alpha_j - J\beta_j$ with $\alpha_j - \sqrt{-1}\beta_j$ being an eigenvector of the eigenvalue $e^{2j\pi\sqrt{-1}/k}$ of P^{-1} for $j \in \mathbb{Z}$. We set

$$W_{P,s} = \{z \in W_P \mid z(t) = \sum_{|j|=(s-1)k+1}^{sk} \exp(\frac{2j\pi t J}{k}) a_j\}, \quad s \in \mathbb{N} \quad (3.8)$$

and

$$W_P^m = \bigoplus_{s=0}^m W_{P,s}. \quad (3.9)$$

Hence the dimension of W_P^0 is exactly $\dim \ker_{\mathbb{R}}(P - I_{2n})$.

We define a quadratic form in W_P^m by

$$\begin{aligned} Q_m(x, y) &= \int_0^1 (\Lambda_l x(t), y(t)) - (C_l^{-1}(t)x(t), y(t)) \\ &= \int_0^1 [(-J\dot{x}(t), y(t)) - (B(t)x(t), y(t))]dt, \quad \forall x, y \in W_P^m, \end{aligned} \quad (3.10)$$

and define a functional $Q_m : W_P^m \rightarrow \mathbb{R}$ by $Q_m(x) = Q_m(x, x)$. We define two linear operators A_l and B_l from W_P^m onto its dual space $(W_P^m)' \cong W_P^m$ by

$$\begin{aligned} \langle A_l x, y \rangle &= \int_0^1 (-J\dot{x}(t) + lx(t), y(t))dt, \\ \langle B_l x, y \rangle &= \int_0^1 ((B(t) + l)x(t), y(t))dt, \quad \forall x, y \in W_P^m. \end{aligned}$$

Since $B(t) + lI_{2n}$ is positive definite, we define $\langle \cdot, \cdot \rangle_m := \langle B_l \cdot, \cdot \rangle$ which is a new inner product in W_P^m . We consider the eigenvalues $\eta_j \in \mathbb{R}$ with respect to the inner $\langle \cdot, \cdot \rangle_m$, that is

$$A_l x_j = \eta_j B_l x_j \quad (3.11)$$

for some $x_j \in W_P^m \setminus \{0\}$. Suppose $\eta_1 \leq \eta_2 \leq \dots \leq \eta_h$ with $h = \dim W_P^m = 2m + \dim \ker_{\mathbb{R}}(P - I)$ (each eigenvalue is counted with multiplicity), the corresponding eigenvectors v_1, \dots, v_h which construct a new basis in W_P^m satisfy

$$\begin{aligned} \langle v_i, v_j \rangle_m &= \delta_{ij}, \\ \langle A_l v_i, v_j \rangle_m &= \eta_i \delta_{ij}, \\ Q_m(v_i, v_j) &= (\eta_i - 1)\delta_{ij}. \end{aligned} \quad (3.12)$$

The Morse indices $m^*(Q_m)$, $*$ = +, 0, - denote the dimension of maximum positive subspace, kernel space and maximum negative subspace of Q_m in W_P^m respectively. By (3.12), we have

$$\begin{aligned} m^+(Q_m) &= \#\{\eta_j \mid 1 \leq j \leq h, \eta_j > 1\}, \\ m^0(Q_m) &= \#\{\eta_j \mid 1 \leq j \leq h, \eta_j = 1\}, \\ m^-(Q_m) &= \#\{\eta_j \mid 1 \leq j \leq h, \eta_j < 1\}. \end{aligned}$$

By Theorem 2.1 and (2.10), for $m > 0$ large enough we have

$$m^-(Q_m) = 2mn + i_P(B), \quad m^0(Q_m) = \nu_P(B). \quad (3.13)$$

We define $Q_{l,m}^* = Q_l^*|_{W_P^m}$ and

$$i_{l,m}^*(B) = m^-(Q_{l,m}^*), \quad \nu_{l,m}^*(B) = m^0(Q_{l,m}^*).$$

By the argument in [12], We have

$$i_{l,m}^*(B) \rightarrow i_l^*(B), \quad \nu_{l,m}^*(B) \rightarrow \nu_l^*(B), \quad \text{as } m \rightarrow \infty.$$

Let $v'_j = A_l v_j$ for $j = 1, 2, \dots, h$. Then it is a basis of W_P^m . It is a basis of W_P^m and it is $Q_{l,m}^*$ -orthogonal, that is

$$Q_{l,m}^*(v'_i, v'_j) = 0, \quad \text{if } i \neq j.$$

Hence $m^-(Q_{l,m}^*)$ equals the number of negative $Q_{l,m}^*(v'_i)$. As a consequence of (3.11) and (3.12), it easily follows that

$$Q_{l,m}^*(v'_i) = \eta_i(\eta_i - 1), \quad (3.14)$$

which is negative if and only if $0 < \eta_i < 1$. If one replaces the inner product $\langle \cdot, \cdot \rangle_m$ by the usual one, that is, one replaces the matrix $B(t) + lI$ by the identity I , the eigenvalues η_j s are replaced by the eigenvalues λ_j s of A_l . It is easy to check that there is a corresponding between the signs of $\{\eta_1, \dots, \eta_h\}$. More precisely, one has

$$\lambda_1 \leq \dots \leq \lambda_r \leq 0 \leq \lambda_{r+1} \quad \text{for some } r \in \{1, \dots, h\} \Leftrightarrow \eta_1 \leq \dots \leq \eta_r \leq 0 \leq \eta_{r+1} \quad (3.15)$$

$$\text{and } \lambda_r = 0 \Leftrightarrow \eta_r = 0. \quad (3.16)$$

So the total multiplicity of negative η_j s equal the total multiplicity of negative λ_j s. But we have

$$\lambda_\kappa = 2\kappa\pi + l, \quad -mk \leq \kappa \leq mk, \quad (3.17)$$

and when $\kappa = 0$, the multiplicity of λ_κ is $\dim \ker_{\mathbb{R}}(P - I)$. Besides, the total multiplicity of λ_κ , $\pm\kappa \in [(s-1)k+1, sk]$ is $2n$ for $1 \leq s \leq m$. Suppose the total multiplicity of the negative λ_κ is M , it is determined by m and l and independent of $B(t)$. From the above argument, we have the following estimation:

$$2n(m-1 - \lfloor \frac{[l/2\pi]}{k} \rfloor) \leq M \leq 2n(m - \lfloor \frac{[l/2\pi]}{k} \rfloor).$$

Hence the total multiplicity of $\lambda_\kappa \in (0, 1)$ is $m^-(Q_m) - M$, and by definition,

$$i_{l,m}^*(B) = m^-(Q_m) - M = 2mn + i_P(B) - M \quad (3.18)$$

for $m > 0$ large enough. \square

Corollary 3.4. *Under the condition of Theorem 2.5, there holds*

$$I_P(B_0, B_1) = i_l^*(B_1) - i_l^*(B_0) \quad \text{for } l > 0 \quad \text{such that } B_j(t) + lI > 0. \quad (3.19)$$

Proof. From Theorem 2.5 and Theorem 3.3 we get (3.19). \square

4 Proof of Theorem 1.1

In order to prove Theorem 1.1 we need a lemma. Let E be a Banach space and $f \in C^2(E, \mathbb{R})$. Set $K = \{x \in E \mid f'(x) = 0\}$ and $f_a = \{x \in L_P \mid f(x) \leq a\}$. If $f'(p) = 0$ and $c = f(p)$, we say that p is a critical point of f and c is critical value. Otherwise, we say that $c \in \mathbb{R}$ is a regular value of f . For any $p \in E$, $f''(p)$ is a self-adjoint operator, the Morse index of p is defined as the dimension of the negative space corresponding to the spectral decomposing, and is denoted by $m^-(f''(p))$. We also set $m^0(f''(p)) = \dim \ker f''(p)$.

Lemma 4.1. *Let $f \in C^2(E, \mathbb{R})$ satisfy the (P.S) condition $f'(0) = 0$ and there exists*

$$r \notin [m^-(f''(0)), m^-(f''(0)) + m^0(f''(0))]$$

with $H_q(E, f_a; \mathbb{R}) \cong \delta_{q,r} \mathbb{R}$. Then f at least one nontrivial critical point $u_1 \neq 0$. Moreover, if $m^0(f''(0)) = 0$ and $m^0(f''(u_1)) \leq |r - m^-(f''(0))|$, then f has one more nontrivial critical point $u_2 \neq u_1$.

Proof. Without loss of generality, we can suppose that $H(t, 0) = 0$. By the condition (H_∞) and Remark 2.3, we find that $i_P(B_1) + \nu_P(B_1) \leq i_P(B_2) + \nu_P(B_2)$, so we have $\nu_P(B_1) = 0$. Firstly, we prove that under the conditions (1.2) or (1.3), it holds that

$$i_P(B_1) \notin [i_P(B_0), i_P(B_0) + \nu_P(B_0)].$$

More precisely, under the condition (1.2), there holds

$$i_P(B_1) = i_P(B_1) + \nu_P(B_1) < i_P(B_0), \quad (4.1)$$

and under the condition (1.3), there holds

$$i_P(B_0) + \nu_P(B_0) < i_P(B_1). \quad (4.2)$$

We only prove (4.1), the proof of (4.2) is similar and we omit it here. By the condition (1.2), we have

$$i_P(B_1) + \nu_P(B_1) \leq i_P(B_1 + lI_{2n}) \leq i_P(B_0).$$

We shall prove that

$$i_P(B_1) < i_P(B_1 + lI_{2n}). \quad (4.3)$$

In fact, suppose $\gamma_1(t) \in P(2n)$ is a symplectic path which is the fundamental solution of the linear Hamiltonian system associated with the matrix $B_1(t)$. Since $JB_1(t) = B_1(t)J$, it is easy to verify that $\gamma := \exp(Jlt)\gamma_1(t)$ is the fundamental solution of the linear Hamiltonian systems

$$\dot{z} = J(B_1(t) + lI_{2n})z.$$

Note that we can regard W_P as

$$W_P = \{x \in W^{1/2,2}([0, 1], \mathbb{R}^{2n}) \mid x(t) = \gamma_P(t)\xi(t), \xi(t) \in W^{1/2,2}([0, 1], \mathbb{R}^{2n})\},$$

since P is symplectic orthogonal, P has the form $P = \exp(M_1)$, the matrix M_1 satisfies $M_1^T J + JM_1 = 0$ and $M_1^T + M_1 = 0$. $\gamma_P(t) = \exp(tM_1)$ as is defined in [25, 27]. It has been proved in [25] that

$$i_P(\gamma) - i_P(\gamma_P) = i(\gamma_P(t)^{-1}\gamma(t)) + n, \quad (4.4)$$

where $\gamma = \gamma(t)$ is the fundamental solution of $\dot{z}(t) = JB(t)z(t)$ with $P^T B(t+1)P = B(t)$, $i(\cdot)$ is the Maslov type index(cf.[29]). Set $\tilde{B}_{\gamma_P}(t) = \gamma_P(t)^T J \dot{\gamma}_P(t) + \gamma_P(t)^T B_1(t) \gamma_P(t)$, we know that $\gamma_P(t)^{-1}\gamma_1(t)$ and $\gamma_P(t)^{-1}\gamma(t)$ are the fundamental solution of $\dot{z}(t) = J\tilde{B}_{\gamma_P}(t)z(t)$ and

$\dot{z}(t) = J(\tilde{B}_{\gamma_P}(t) + lI_{2n})z(t)$ respectively. For $l = 2\pi$, the paths $\gamma_P(t)^{-1}\gamma(t)$ and $\gamma_P(t)^{-1}\gamma_1(t)$ have the same endpoint. Moreover, the rotation numbers satisfy

$$\Delta_1(\gamma_P(t)^{-1}\gamma(t)) = 2n + \Delta_1(\gamma_P(t)^{-1}\gamma_1(t)).$$

Then we have

$$i(\gamma_P(t)^{-1}\gamma_1(t)) + 2n \leq i(\gamma_P(t)^{-1}\gamma(t)). \quad (4.5)$$

Finally by (4.4) and (4.5), we get

$$i_P(B_1) + 2n < i_P(B_1 + lI_{2n}). \quad (4.6)$$

By the condition (H_∞) , $H''(t, x)$ is bounded and there exist $\mu_1, \mu > 0$ such that

$$I_{2n} \leq H''(t, x) + \mu I \leq \mu_1 I_{2n}, \quad \forall (t, x). \quad (4.7)$$

We define a convex function $N(t, x) = H(t, x) + \frac{1}{2}\mu|x|^2$. Its Fenchel dual $N^*(t, x)$ which is defined by

$$N^*(t, x) = \sup_{y \in \mathbb{R}^{2n}} \{(x, y) - N(t, x)\}$$

satisfying (cf.[10])

$$\begin{aligned} N^*(t, x) &\in C^2(\mathbb{R} \times \mathbb{R}^{2n}) \\ N^{*''}(t, y) &= N''(t, x)^{-1}, \quad \text{for } y = N'(t, x). \end{aligned} \quad (4.8)$$

From (4.7) we have

$$\mu_1^{-1}I_{2n} \leq N^{*''}(t, y) \leq I_{2n}, \quad \forall (t, y). \quad (4.9)$$

So we have $|x| \rightarrow \infty$ if and only if $|y| \rightarrow \infty$ with $y = N'(t, x)$. From the condition (H_∞) and (4.8), there exists r_1 such that

$$(B_2(t) + \mu I_{2n})^{-1} \leq N^{*''}(t, y) \leq (B_1(t) + \mu I_{2n})^{-1}, \quad \forall (t, y) \text{ with } |y| \geq r_1. \quad (4.10)$$

We choose $\mu > 0$ satisfying (4.7) and $\mu \notin \sigma(A)$. We recall that $(\Lambda_\mu x)(t) = -J\dot{x}(t) + \mu x(t)$. Consider the functional defined by

$$f(u) = - \int_0^1 [\frac{1}{2}(\Lambda_\mu^{-1}u(t), u(t)) - N^*(t, u(t))]dt, \quad \forall u \in L_P, \quad (4.11)$$

it is easy to see that $f \in C^2(L_P, \mathbb{R})$. Next we prove that f satisfies the Palais-Smale condition(cf.[6, 10]).

Assume that $\{u_j\}$ is a sequence in W_P such that $f(u_j)$ is bounded and $f'(u_j) \rightarrow 0$. By (H_0) , we have $N'(t, 0) = 0$ and $N^{*'}(t, 0) = 0$ and

$$(f'(u), v) = - \int_0^1 [(\Lambda_\mu^{-1}u(t), v(t)) - (N^{*'}(t, u(t)), v(t))]dt, \quad \forall u, v \in L_P. \quad (4.12)$$

Note that $\int_0^1 N^{*''}(t, \tau u(t))d\tau u(t) = N^{*'}(t, u(t))$, we have

$$\Lambda_\mu^{-1}u_j(t) - \int_0^1 N^{*''}(t, \tau u_j(t))d\tau u_j(t) \rightarrow 0, \quad \text{in } L_P. \quad (4.13)$$

If $\|u_j\|_2 \rightarrow \infty$, we set $x_j = u_j/\|u_j\|_2$. L_P is a reflexive Hilbert space and $\|x_j\|_2 = 1$, $\forall j \in \mathbb{N}$, without loss of generality, we assume $x_j \rightharpoonup x_0$, and hence $\Lambda_\mu^{-1}x_j \rightarrow \Lambda_\mu^{-1}x_0$ in L_P . For any $\delta \in (0, 1)$ fixed, set

$$C_j(t) = \begin{cases} \int_0^1 N^{*''}(t, \tau u_j(t)) d\tau, & \text{if } |u_j| \geq r_1/\delta, \\ (B_1(t) + \mu I_{2n})^{-1}, & \text{otherwise,} \end{cases}$$

$$\eta_j(t) = \int_0^1 N^{*''}(t, \tau u_j(t)) d\tau u_j(t) - C_j(t)u_j(t).$$

Then there exists a constant $M_1 > 0$ such that

$$|\eta_j| \leq M_1 \quad \text{for a.e. } t \in (0, k) \quad (4.14)$$

and

$$(1 - \delta)(B_2(t) + \mu I_{2n})^{-1} + \delta I_{2n} \leq C_j(t) \leq (1 - \delta)(B_1(t) + \mu I_{2n})^{-1} + \mu_1^{-1} \delta I_{2n}.$$

So for every $s > 0$, there exists $\delta > 0$ such that

$$((B_2(t) + sI_{2n}) + \mu I_{2n})^{-1} \leq C_j(t) \leq ((B_1(t) - sI_{2n}) + \mu I_{2n})^{-1}, \quad \forall t \in (0, k). \quad (4.15)$$

Now we may assume $C_j^{-1}u(t) \rightharpoonup B_0(t)u(t)$ in L_P for $u \in L_P$ with $\mu I + B_1 - \varepsilon I \leq B_0 \leq \mu I_{2n} + B_2 + sI_{2n}$. Let $\Lambda_\mu^{-1}x_0(t) = y_0(t)$, from (4.13)-(4.15), we have

$$Jy_0(t) + (B_0(t) - \mu I_{2n})y_0(t) = 0, \quad y_0(1) = Py_0(0). \quad (4.16)$$

From the condition (H_∞) and Theorem 2.8 (2), for $s > 0$ small enough, we have $\nu_P(B_1 - sI_{2n}) = \nu_P(B_2 + sI_{2n}) = 0$ and $i_P(B_1 - sI_{2n}) = i_P(B_2 + sI_{2n})$. So $\nu_P(B_0 - \mu I_{2n})$ vanishes. This is impossible since $\|y_0\|_2 = 1$ and y_0 is a nontrivial solution of (4.16). Hence $\|u_j\|_2$ is bounded.

Assume $u_j \rightharpoonup u_0$ in L_P , then $\Lambda_\mu^{-1}u_j \rightarrow \Lambda_\mu^{-1}u_0$. Let $\zeta_j := \Lambda_\mu^{-1}u_j - N^{*'}(t, u_j(t))$, then $N^{*'}(t, u_j(t)) = \Lambda_\mu^{-1}u_j - \zeta_j \rightarrow \Lambda_\mu^{-1}u_0$ by (4.13). The Fenchel conjugate formula gives $u_j = N'(\Lambda_\mu^{-1}u_j - \zeta_j) \rightarrow N'(\Lambda_\mu^{-1}u_0)$. So f satisfies the (P.S) condition.

There is a one-to-one correspondence from the critical points of f to the solutions of the systems (1.1). Note that 0 is a trivial critical point of f and $N^{*'}(t, 0) = 0$. At every critical point u_0 , the second variation of f defines a quadratic form on L_P by

$$(f''(u_0)u, u) = - \int_0^k [(\Lambda_\mu^{-1}u(t), u(t)) - (N^{*''}(t, u_0(t))u(t), u(t))], \quad \forall u \in L_P. \quad (4.17)$$

The critical point u_0 corresponds to a solution $x_0 = \Lambda_\mu^{-1}u_0(t)$. By (4.8), we have

$$N^{*''}(t, u_0(t)) = N''(t, x_0(t))^{-1} = (H''(t, x_0) + \mu I_{2n})^{-1}. \quad (4.18)$$

By definition, we have

$$m^-(f''(u_0)) = i_\mu^*(B), \quad m^0(f''(u_0)) = \nu_\mu^*(B), \quad \text{where } B(t) = H''(t, x_0). \quad (4.19)$$

By Theorem 3.3, we have

$$\nu_\mu^*(B) = \nu_P(B), \quad i_\mu^*(B) = 2mn + i_P(B) - M,$$

The index pair $(i_P(B), \nu_P(B))$ is the Maslov P -index of the linear Hamiltonian system

$$\dot{y}(t) = JB(t)y(t).$$

By condition (1.2) and the result (4.6), we have

$$i_P(B_1) + \nu_P(B_1) + 2n < i_P(B_0). \quad (4.20)$$

By condition (1.3), similarly we have

$$i_P(B_0) + \nu_P(B_0) + 2n < i_P(B_1). \quad (4.21)$$

From (4.20)-(4.21), Theorem 2.5 and Corollary 3.4, we get that

$$|i_P(B_0) - i_P(B_1)| \geq 2n \quad \text{and} \quad |i_\mu^*(B_0) - i_\mu^*(B_1)| \geq 2n. \quad (4.22)$$

Note that

$$N^{*''}(t, 0) = N''(t, 0)^{-1} = (H''(t, 0) + \mu I)^{-1}.$$

and $B_0(t) = H''(t, 0)$, so

$$m^-(f''(0)) = i_\mu^*(B_0), \quad m^0(f''(0)) = \nu_\mu^*(B_0).$$

Hence, by Lemma 4.1, we only need to show the homology groups satisfy

$$H_q(L_P, f_a; \mathbb{R}) \cong \delta_{q,r} \mathbb{R}, \quad q = 0, 1, 2, \dots, \quad (4.23)$$

for some $a \in \mathbb{R}$ and $r = i_\mu^*(B_1)$. $f_a = \{x \in L_P \mid f(x) \leq a\}$ is the level set below a . We proceed in three steps.

Step 1. For $P \in Sp(2n)$ with $P^T P = I_{2n}$, $B_j(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$ satisfies $P^T B_j(t+1)P = B_j(t)$, $j = 1, 2$ and $B_1(t) < B_2(t)$, there holds

$$L_P = L_\mu^-(B_1) \oplus L_\mu^+(B_2),$$

where L_μ^* for $* = \pm, 0$ is defined in Section 3.

In fact, if $0 \neq u \in L_\mu^-(B_1)$, then $Q_{l, B_1}^*(u) < 0$,

$$Q_{l, B_2}^*(u) \leq Q_{l, B_1}^*(u) < 0,$$

and $u \notin L_\mu^+(B_2)$. We only need to prove that $L_P = L_\mu^-(B_1) + L_\mu^+(B_2)$.

By Theorem 3.3, $\nu_\mu^* = \nu_P(B_2) = 0$, we have $L_P = L_\mu^-(B_2) \oplus L_\mu^+(B_2)$. By (H_∞) and Corollary 3.4, we have $i_\mu^*(B_1) = i_\mu^*(B_2) = r$. Suppose $\xi_1, \xi_2, \dots, \xi_r$ be a basis of $L_\mu^-(B_1)$. We have decompositions $\xi_j = \xi_j^- + \xi_j^+$ with $\xi_j^- \in L_\mu^-(B_2)$ and $\xi_j^+ \in L_\mu^+(B_2)$. It is clear that $\{\xi_j^-\}_{j=1}^r$ is linear independent. If $\sum_{j=1}^r \alpha_j \xi_j^- = 0$, then $\bar{x} := \sum_{j=1}^r \alpha_j \xi_j = \sum_{j=1}^r \alpha_j \xi_j^+ \in L_\mu^+(B_2)$, and $\bar{x} \in L_\mu^-(B_1)$, so $\bar{x} = 0$ and $\alpha = 0, j = 1, \dots, j$.

Since $\dim L_\mu^-(B_2) = i_\mu^*(B_2) = i_\mu^*(B_1) = r$, $\{\xi_j^-\}_{j=1}^r$ is a basis of $L_\mu^-(B_2)$. For any $\xi \in L_P$ written as $\xi = \xi^- + \xi^+$ with $\xi^- \in L_\mu^-(B_2)$ and $\xi^+ \in L_\mu^+(B_2)$, we have $\xi^- = \sum_{j=1}^r \beta_j \xi_j^-$. So

$$\xi = \sum_{i=1}^r \beta_j \xi_j + (u^+ - \sum_{i=1}^r \beta_j \xi_j^+),$$

the first sum lies in $L_\mu^-(B_1)$ and the remainder is in $L_\mu^+(B_2)$.

Step 2. For sufficiently small $s > 0$, we set $D_R := L_\mu^-(B_1 - sI_{2n}) \oplus \{L_\mu^+(B_2 + sI_{2n}) \mid \|u\| \leq R\}$. For $R > 0$ and $-a > 0$ large enough, we have the following deformation result:

$$H_q(L_P, f_a; \mathbb{R}) = H_q(D_R, D_R \cap f_a; \mathbb{R}), \quad \text{for } q = 0, 1, 2, \dots \quad (4.24)$$

In fact, from the condition (H_∞) and Theorem 2.8, we have $\nu_P(B_1 - sI_{2n}) = \nu_P(B_1) = 0$, $\nu_P(B_2 + sI_{2n}) = \nu_P(B_2) = 0$ and then $i_P(B_1 - sI_{2n}) = i_P(B_1) = i_P(B_2) = i_P(B_2 + sI)$.

By the condition (H_∞) and Step 1, any $u \in L_P$ can be written as $u = u_1 + u_2$ with $u_1 \in L_\mu^-(B_1 - sI_{2n})$ and $u_2 \in L_\mu^+(B_2 + sI_{2n})$, from (4.12), we have

$$\begin{aligned} (f'(u), u_2 - u_1) &= - \int_0^k [(\Lambda_\mu^{-1}u, u_2 - u_1) - (N^{*'}(t, u(t)), u_2 - u_1)] dt \\ &= \int_0^k (\Lambda_\mu^{-1}u_1, u_1) dt - \int_0^k \left(\int_0^1 N^{*''}(t, \tau u(t)) d\tau u_1, u_1 \right) dt - \int_0^k (\Lambda_\mu^{-1}u_2, u_2) dt \\ &\quad - \int_0^k \left(\int_0^1 N^{*''}(t, \tau u(t)) d\tau u_2, u_2 \right) dt. \end{aligned} \quad (4.25)$$

By (4.9) and (4.10), we have

$$\begin{aligned} \int_0^k \left(\int_0^1 N^{*''}(t, \tau u(t)) d\tau u_1, u_1 \right) dt &= \int_0^k \left(\int_0^{h(t, u)} N^{*''}(t, \tau u(t)) d\tau u_1, u_1 \right) dt + \int_0^k \left(\int_{h(t, u)}^1 N^{*''}(t, \tau u(t)) d\tau u_1, u_1 \right) dt \\ &\leq c_0 \|u\|_2 + \int_0^k (B_1(t) + \mu I_{2n} - sI_{2n})^{-1} u_1, u_1 dt, \end{aligned} \quad (4.26)$$

where $h(t, u) = r_1/|u(t)|$. Similarly, we have

$$\begin{aligned} \int_0^k \left(\int_0^1 N^{*''}(t, \tau u(t)) d\tau u_2, u_2 \right) dt &\geq \int_0^k \int_{h(t, u)}^1 N^{*''}(t, \tau u(t)) d\tau u_2, u_2 dt \\ &\geq \int_0^k (B_2(t) + \mu I_{2n} + sI_{2n})^{-1} u_2, u_2 dt - c \|u\|_2, \quad \text{for } c > 0. \end{aligned} \quad (4.27)$$

Note that in the subspace $L_\mu^-(B_1 - sI_{2n})$ of L_P , the norm $\|\cdot\|_2$ is equivalent to $\|\cdot\|_1$ defined by

$$\|\cdot\|_1 := \left(\int_0^k (B_1(t) + \mu I_{2n} - sI)^{-1} u_1, u_1 dt \right)^{1/2}.$$

In this way, by (4.25)-(4.27) we obtain

$$(f'(u), u_2 - u_1) \geq c_1 \|u_1\|_2^2 + c_2 \|u_2\|_2^2 - c_3 (\|u_1\|_2 + \|u_2\|_2). \quad (4.28)$$

Thus, for large R with $\|u_1\|_2 \geq R$ or $\|u_2\|_2 \geq R$, we have

$$-(f'(u), u_2 - u_1) < -1. \quad (4.29)$$

We know from (4.29) that f has no critical point outside D_R and $-f'(u)$ points inwards to D_R on ∂D_R . Therefore, we can define the deformation by negative flow. For any $u = u_1 + u_2 \notin D_R$, let $\sigma(t, u) = e^\theta u_1 + e^{-\theta} u_2$, and $d_u = \ln \|u_2\|_2 - \ln R$. We define the deformation map $\eta : [0, 1] \times L_P \rightarrow L_P$ by

$$\eta(t, u_1 + u_2) = \begin{cases} u_1 + u_2, & \text{if } \|u_2\|_2 \leq R, \\ \sigma(d_u \theta, u), & \text{if } \|u_2\|_2 > R. \end{cases}$$

Then η is continuous and satisfies

$$\eta(0, \cdot) = id, \quad \eta(1, L_P) \subset D_R, \quad \eta(1, f_a) \subset D_R \cap f_a,$$

$$\eta(\theta, f_a) \subset f_a, \quad \eta(\theta, \cdot) |_{D_R} = id |_{D_R}.$$

Hence the pair $(D_R, D_R \cap f_a)$ is a deformation retract of the pair (L_P, f_a) .

Step 3. For $R, -a > 0$ large enough, there holds

$$H_q(D_R, D_R \cap f_a; \mathbb{R}) \cong \delta_{q,r} \mathbb{R}, \quad q = 0, 1, 2, \dots, \quad (4.30)$$

In fact, similarly to the above computation, for a large number $m > 0$, we have

$$\begin{aligned} & \int_0^k N^*(t, u(t)) dt \\ &= \int_0^k \left(\int_0^1 \tau d\tau \int_0^1 (N^{*''}(t, \tau su(t)) dsu(t), u(t)) \right) dt + \int_0^k N^*(t, 0) dt \\ &\leq \int_{|u(t)| \geq mr_1} \left(\int_0^1 \tau d\tau \int_0^1 (N^{*''}(t, \tau su(t)) dsu(t), u(t)) \right) dt + c_m \\ &\leq \int_{|u(t)| \geq mr_1} \left(\int \int_{|\tau su(t)| \geq r_1, \tau, s \in [0, 1]} \tau N^{*''}(t, \tau su(t)) ds d\tau u(t), u(t) \right) dt \\ &\quad + \int_{|u(t)| \geq mr_1} \left(\int \int_{|\tau su(t)| \leq r_1, \tau, s \in [0, 1]} \tau N^{*''}(t, \tau su(t)) ds d\tau u(t), u(t) \right) dt + c_m \\ &\leq \frac{1}{2} \int_0^k (B_1(t) + \mu I_{2n})^{-1} u(t), u(t) dt + d_m \|u_2\|_2 + c_m, \end{aligned}$$

where c_m and d_m are constants depending only on m and $d_m \rightarrow 0$ as $m \rightarrow \infty$. Hence for the small s in Step 2 above, we can choose a large number m such that

$$\int_0^k N^*(t, u(t)) dt \leq \frac{1}{2} \int_0^k (B_1(t) + \mu I_{2n} - sI)^{-1} u(t), u(t) dt + C \quad \forall u \in L_P$$

for some constant $C > 0$. Together with (4.11), this yields, for any $u = u_1 + u_2$ with $u_1 \in L_\mu^-(B_1 - sI_{2n})$ and $u_2 \in L_\mu^+(B_2 + sI_{2n})$ with $\|u_2\|_2 \leq R$, we have

$$f(u) \leq -C_1 \|u_1\|_2^2 + C_2 \|u_1\|_2 + C_3,$$

where C_j , $j = 1, 2, 3$ are constants and $C_1 > 0$. It implies that $f(u) \rightarrow -\infty$ if and only if $\|u_1\|_2 \rightarrow \infty$ uniformly for $u_2 \in L_\mu^+(B_2 + sI_{2n})$ with $\|u_2\|_2 \leq R$. In the following we denote $B_r = \{x \in L_P \mid \|x\|_2 \leq r\}$ the ball with radius r in L_P . Thus there exist $T > 0$, $a_1 < a_2 < -T$, and $R < R_1 < R_2 < R_3$ such that

$$\begin{aligned} & (L_\mu^+(B_2 + sI_{2n}) \cap B_{R_3} \oplus (L_\mu^-(B_1 - sI_{2n}) \setminus B_{R_2}) \subset f_{a_1} \cap D_{R_3} \\ & \subset (L_\mu^+(B_2 + sI_{2n}) \cap B_{R_3} \oplus (L_\mu^-(B_1 - sI_{2n}) \setminus B_{R_1}) \subset f_{a_2} \cap D_{R_3}. \end{aligned}$$

For any $u \in D_{R_3} \cap (f_{a_2} \setminus f_{a_1})$, since $\sigma(t, u) = e^\theta u_1 + e^{-\theta} u_2$, the function $f(\sigma(t, u))$ is continuous in t and satisfies $f(\sigma(\theta, u)) = f(u) > a_1$ and $f(\sigma(t, u)) \rightarrow -\infty$ as $t \rightarrow +\infty$. It implies that there exists $\theta_0 = \theta_0(u) > 0$ such that $f(\sigma(\theta_0, u)) = a_1$. But by (4.29),

$$\frac{d}{d\theta} f(\sigma(t, u)) \leq -1, \quad \text{at any point } \theta > 0.$$

By the implicit function theorem, $\theta_0(u)$ is continuous in u . We define another deformation map $\eta_0 : [0, 1] \times f_{a_2} \cap D_{R_3} \rightarrow f_{a_2} \cap D_{R_3}$ by

$$\eta_0(\theta, u) = \begin{cases} u, & \text{if } f_{a_1} \cap D_{R_3}, \\ \sigma(\theta_0(u), u), & \text{if } u \in D_{R_3} \cap (f_{a_2} \setminus f_{a_1}). \end{cases}$$

It is clear that η_0 is a deformation from $f_{a_2} \cap D_{R_3}$ to $f_{a_1} \cap D_{R_3}$. We now define

$$\tilde{\eta}(u) = d(\eta_0(1, u)) \quad \text{with} \quad d(u) = \begin{cases} u, & \|u_1\|_2 \geq R_1, \\ u_2 + \frac{u_1}{\|u_1\|_2} R_1, & 0 < \|u_1\|_2 < R_1. \end{cases}$$

This map defines a strong deformation retract:

$$\tilde{\eta} : D_{R_3} \cap f_{a_2} \rightarrow L_\mu^+(B_2 + sI_{2n}) \cap B_{R_3} \oplus (L_\mu^-(B_1 - sI_{2n}) \cap \{u \in L_P \mid \|u\|_2 \geq R_1\}).$$

Now we can compute the homology groups

$$\begin{aligned} H_q(D_{R_3}, D_{R_3} \cap f_{a_2}; \mathbb{R}) & \cong H_q(D_{R_3}, L_\mu^+(B_2 + sI_{2n}) \cap B_{R_3} \oplus (L_\mu^-(B_1 - sI_{2n}) \cap \{u \in L_P \mid \|u\|_2 \geq R_1\}); \mathbb{R}) \\ & \cong H_q(L_\mu^-(B_1 - sI_{2n}) \cap B_{R_3}, \partial(L_\mu^-(B_1 - sI_{2n}) \cap B_{R_3}); \mathbb{R}) \\ & \cong \delta_{q,r} \mathbb{R}. \end{aligned}$$

□

Remark 4.2. The method of the proof (4.23) comes from [5], but we have modified it to suit our case.

Corollary 4.3. Suppose that $P \in Sp(2n)$ satisfies $P^T P = I_{2n}$, H satisfies conditions (H) , (H_0) , (H_∞) . Suppose $B_0(t) = H''(t, 0)$ satisfying one of the following twisted conditions:

- (I) $B_1(t) < B_0(t)$, there exists $\lambda \in (0, 1)$ such that $\nu_P((1 - \lambda)B_1 + \lambda B_0) \neq 0$;
- (II) $B_0(t) < B_1(t)$, there exists $\lambda \in (0, 1)$ such that $\nu_P((1 - \lambda)B_0 + \lambda B_1) \neq 0$.

Then the system (1.1) possesses at least one non-trivial P -solution. Furthermore, if $\nu_P(B_0) = 0$ and in, we replace the second condition by

$$\sum_{\lambda \in (0,1)} \nu_P((1-\lambda)B_1 + \lambda B_0) \geq 2n,$$

or in , we replace the second condition by

$$\sum_{\lambda \in (0,1)} \nu_P((1-\lambda)B_0 + \lambda B_1) \geq 2n.$$

Then the system (1.1) possesses at least two non-trivial P -solutions.

Proof. The proof follows from Lemma 2.7, Lemma 4.1 and the above proof of Theorem 1.1. In the first case, we have $r = i_P(B_1) \notin [i_P(B_0), i_P(B_0) + \nu_P(B_0)]$. In the second case we have $|i_P(B_0) - i_P(B_1)| \geq 2n$. □

The proof of Theorem 1.1 in fact proves the following result.

Theorem 4.4. *Suppose that $P \in Sp(2n)$ satisfies $P^T P = I_{2n}$, H satisfies conditions (H) , (H_0) , (H_∞) . Suppose $B_0(t) = H''(t, 0)$ satisfying the following twisted condition:*

$$i_P(B_1) \notin [i_P(B_0), i_P(B_0) + \nu_P(B_0)]. \quad (4.31)$$

Then the system (1.1) possesses at least one non-trivial P -solution. Furthermore, if $\nu_P(B_0) = 0$ and $|i_P(B_1) - i_P(B_0)| \geq 2n$, the system (1.1) possesses at least two non-trivial P -solutions.

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